A pesky hypothesis for the integral test and the comparison tests:

$a_n \geq 0$

↳ not all series have positive terms

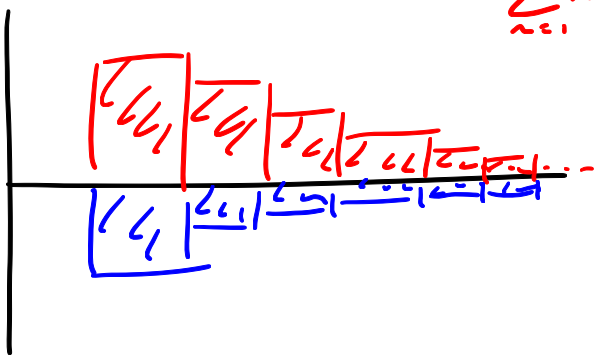
converges

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Ex 1 Does $\sum_{n=1}^{\infty} \frac{-1}{n^2+1}$ converge or diverge?

(a) Converges

(b) Diverges



$$\sum_{n=1}^{\infty} \frac{-1}{n^2+1} = -\sum_{n=1}^{\infty} \frac{1}{n^2+1} \approx -1.7\ldots$$

$\sum_{n=1}^{\infty} \frac{-1}{n^2+1}$

If all terms are negative, factor out -1

$= -1 + \cdots = -(1 + \cdots)$

Ex 2 Can you apply either comparison test to $\sum_{n=1}^{\infty} \frac{n^2-7}{n^4+5}$?

(a) yes

(b) no

eventually

You might think "no" because...

- Comparison test requires $a_n \geq 0$ and for $n=1$, $\frac{n^2-7}{n^4+5} = \frac{-6}{6} = -1$

Q: Can we factor out -1?

- No, because $n=3$, $\frac{n^2-7}{n^4+5} = \frac{9}{86}$, so factoring out -1 makes that term negative

Why is the answer actually "yes"?

- eventually all terms are positive

D.C.T: $0 \le \frac{n^2-7}{n^4+5} \le \frac{n^2}{n^4+5} \le \frac{n^2}{n^4} = \frac{1}{n^2}$

for $n \ge 3$

$\longrightarrow \sum_{n=1}^{\infty} \frac{n^2-7}{n^4+5} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\Longrightarrow$ By the D.C.T. $\sum_{n=1}^{\infty} \frac{n^2-7}{n^4+5}$ converges

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Case: $+ \quad + + \quad - \quad - \quad - \quad - \quad - \quad \cdots -$

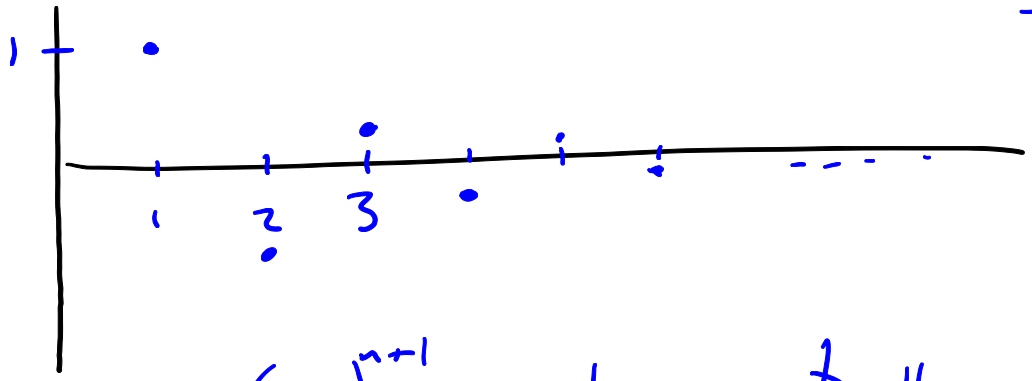
↳ factor out -1: $-(- \quad - - \quad + + + + + \cdots)$

Case: $+ \; - \; + \; - \; + \; - \; + \; - \; + \; - \; + \; - \; \cdots$

↳ alternating

Ex 3 Can you apply the integral test or either comparison test to $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$? What are the partial sums of this series?

$\frac{(-1)^{n+1}}{n}$

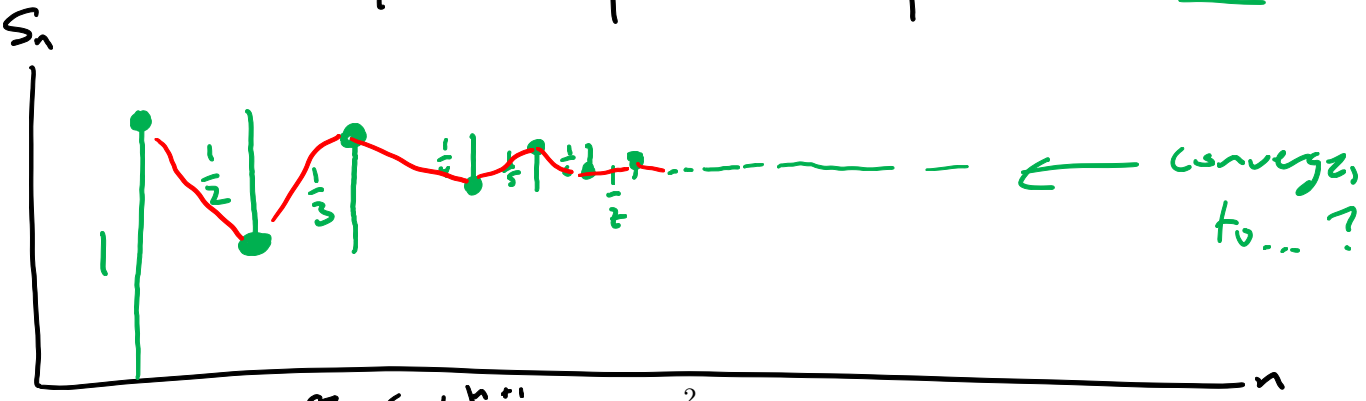


→ $\frac{(-1)^{n+1}}{n}$ not eventually positive or eventually negative

→ no integral or comparison tests

Partial Sums

n	1	2	3	4	5
$\frac{(-1)^{n+1}}{n}$	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$
partial sum $= S_n$	1	$1-\frac{1}{2}=\frac{1}{2}$	$1-\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	$\cdots-\frac{1}{4}=\frac{7}{12}$	$\frac{47}{60}$



→ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges

Def: An alternating series is any series of the form

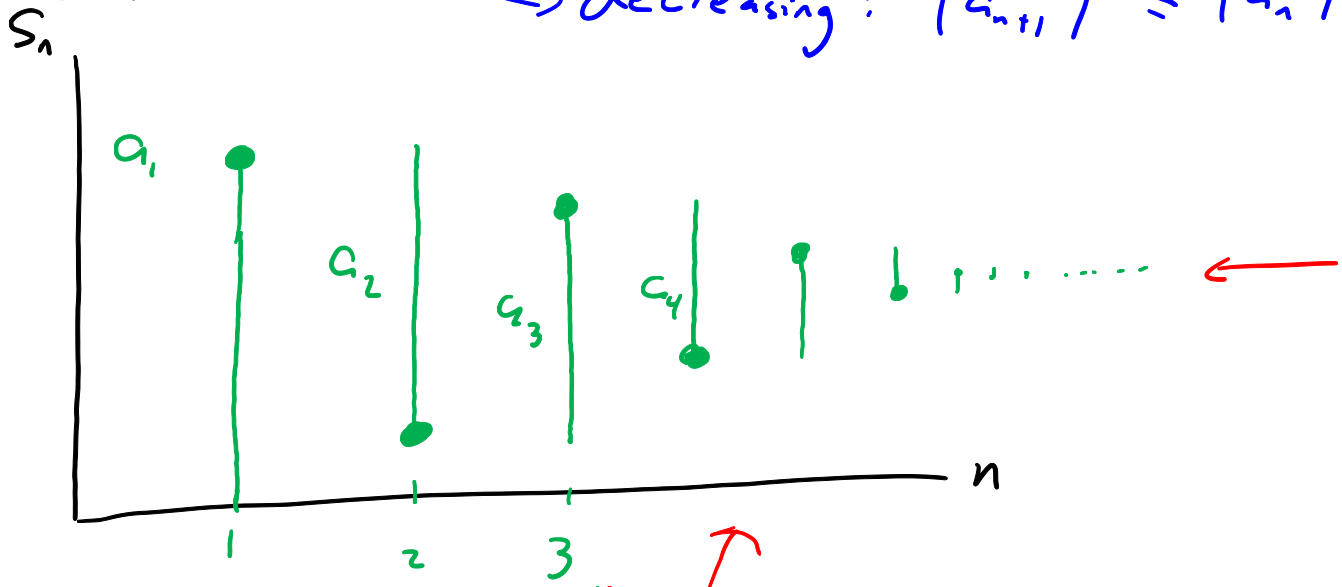
$$\sum_{n=n_0}^{\infty} (-1)^n b_n \quad \text{OR} \quad \sum_{n=n_0}^{\infty} (-1)^{n+1} b_n$$

where $b_n > 0 \quad \forall n \geq n_0$

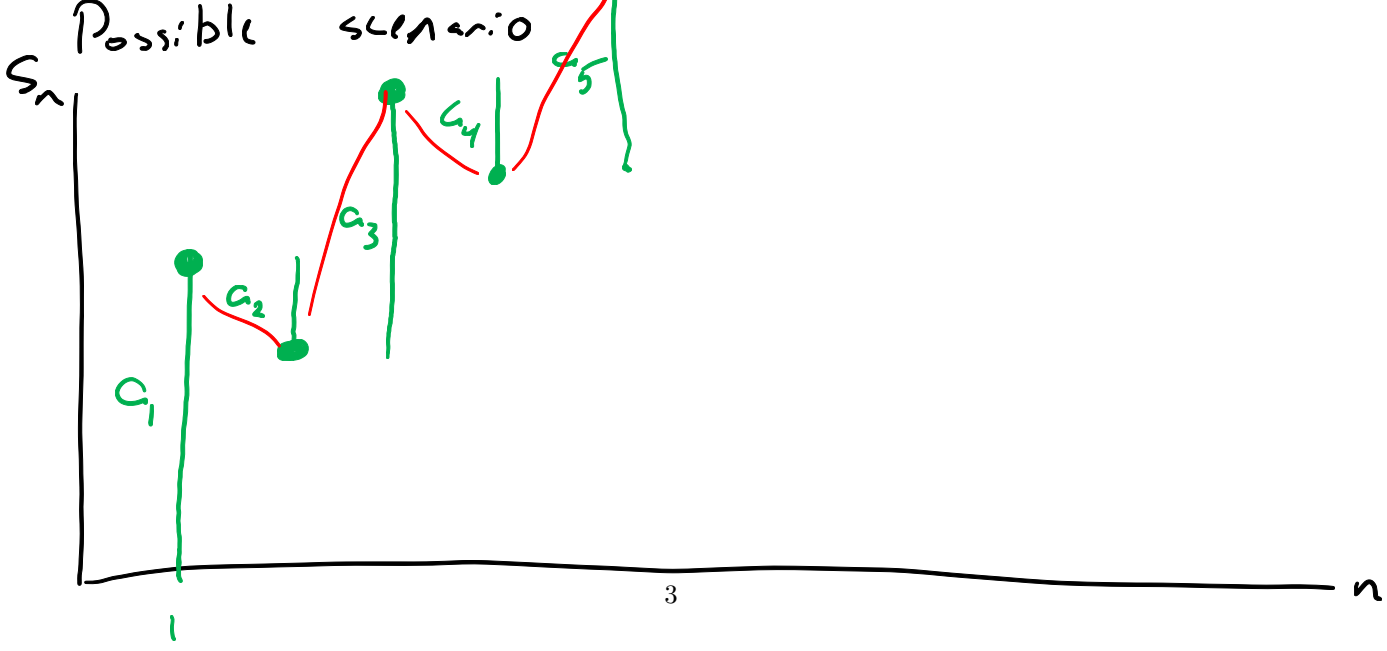
Thm: (Alternating Series Test) If a series $\sum_{n=n_0}^{\infty} a_n$ is alternating AND the sequence $\{|a_n|\}_{n=n_0}^{\infty}$ converges monotonically to 0, then the series converges.

Q. Why is this true?

monotonically decreasing: $|a_{n+1}| \leq |a_n|$



Q. What if $|a_n|$ isn't monotone?



Monotone: either eventually (always) decreasing or eventually (always) increasing

Ex 4 Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges.

Alt. Series: $\sum (-1)^{??} b_n$, $b_n > 0$

$(-1)^{n+1} \cdot \frac{1}{n^2}$, $b_n = \frac{1}{n^2} > 0$ ✓

$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \lim_{n\to\infty} \frac{1}{n^2} = 0$ ✓

Want to check: $|a_{n+1}| \le |a_n|$

LHS: $\left|\frac{(-1)^{n+2}}{(n+1)^2}\right| = \frac{1}{(n+1)^2}$

RHS: $\left|\frac{(-1)^{n+1}}{n^2}\right| = \frac{1}{n^2}$

$\frac{1}{(n+1)^2} \le \frac{1}{n^2}$ ✓

$\Rightarrow$ The series converges by the A.S.T.

→ Alternate: $|a_n| = \frac{1}{n^2}$ decreasing ✓

Alternate 2: $f(x) = \frac{1}{x^2}$

$f'(x) = \frac{-1}{x^3} < 0$ for $x > 0$

$\Rightarrow$ f decreasing for $x > 0$

Ex 5 Can we use the alternating series test for $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ when $p \neq 0$?

(a) Yes
(b) No

AST requires $\lim_{n\to\infty} |a_n| = 0$

Ex: $p = -1$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{-1}} = \sum_{n=1}^{\infty} (-1)^{n+1} n$

$\lim_{n\to\infty} |(-1)^{n+1} n| = \lim_{n\to\infty} n = \infty$

$\Rightarrow$ A.S.T. does not apply

Want to show: $|a_n|$ is monotone

- eventually decreasing
- eventually increasing

$$|a_n| = \left| \frac{(-1)^{n+1}}{n^2} \right| = \frac{1}{n^2}$$

as n increases a_n decreases

As n increases: $\frac{1}{n^2}$ ← remains constant, n^2 ← increases → decreasing ✓

Ex: As n increases: $ne^{-n^2} = \frac{n}{e^{n^2}}$ ← increasing, ← increasing → ?

$$f(x) = xe^{-x^2}$$

$$f'(x) = e^{-x^2} + x e^{-x^2} \cdot (-2x)$$

$$= e^{-x^2}(1 - 2x^2) < 0 \quad \text{for } x \geq 1$$

Q. Can we estimate the remainder of a convergent alternating series?

$$\sum_{n=n_0}^{\infty} (-1)^n b_n \qquad b_n \geq 0$$

$$R_k = b_{k+1} - b_{k+2} + b_{k+3} - b_{k+4} + \cdots$$

$$= b_{k+1} - (b_{k+2} - b_{k+3}) - (b_{k+4} - b_{k+5}) - (\,) \cdots$$

$(b_{k+2} - b_{k+3}) \geq 0$, $(b_{k+4} - b_{k+5}) \geq 0$, $(\,) \geq 0$

$$\leq b_{k+1}$$

$$\rightarrow |R_k| \leq |b_{k+1}|$$

Ex 6 Estimate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ with error $< 10^{-5}$.

Want: $R_k < 10^{-5}$

Goal: $\left| \frac{(-1)^{k+1}}{k+1} \right| < 10^{-5} \iff \frac{1}{k+1} < 10^{-5}$

$$\iff \frac{10^5}{k+1} < 1$$

$$\iff 10^5 < k+1$$

$$\iff 10^5 - 1 < k$$

Take: $k = 10^5 \rightarrow |R_{100000}| < 10^{-5}$

$$S_{100000} \approx 0.693$$

Def: A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges we say $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Thm: (Absolute Convergence Test) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof: Unenlightening

$\sum_{n=1}^{\infty} |a_n|$ converges

Note:

The converse of the A.C.T. is not true:

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges (by A.S.T) but $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Ex 7 Show that $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges.

$\Rightarrow$ can't use A.S.T. here

↳ $\left|\frac{\sin(n)}{n^2}\right|$ not monotone

↳ $\frac{\sin(n)}{n^2}$ doesn't alternate

n	1	2	3	4	5	6	7
$\frac{\sin(n)}{n^2}$	+	+	+	−	−	−	+

$$\frac{\sin(n)}{n^2} \approx \frac{1}{n^2}$$

$$0 \le \left| \frac{\sin(n)}{n^2} \right| \le \frac{1}{n^2}$$

Note that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

By *D.C.T. $\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^2} \right|$ converges

This means that $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges absolutely.

By *A.C.T. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges.

L.C.T.

$$\frac{3n^2 + 2n + 1}{4n^7 - 3} \approx \frac{3n^2}{4n^7} = \frac{3}{4} \cdot \frac{1}{n^5}$$

$$\sin(n) \approx 1 \approx \cos(n)$$

need: $\sin(n) + 7$

$$2^n \approx 2^n \not\approx 3^n \quad \frac{n}{n^5} \Rightarrow \frac{\log(n)}{n^5} \not\approx \frac{1}{n^5}$$

$$\log(n) \ll n^{\varepsilon} \quad \frac{1}{n^4}$$

Note: Infinite sums are WEIRD.

- If $\sum_{n=1}^{\infty} a_n$ converges absolutely, you can rearrange the terms in the sum and the value remains constant
- If $\sum_{n=1}^{\infty} a_n$ converges conditionally, rearranging terms can change the value!

$\sum_{k=1}^{\infty} \frac{1}{2k-1}$ diverges to ∞

Ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

↳ converges

Find k s.t.

$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2k-1} > 10$

Now subtract $\frac{1}{2}$

$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} - \frac{1}{2} > 9.5$

Find l s.t.

$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} - \frac{1}{2} + \frac{1}{2(k+1)-1} + \cdots + \frac{1}{2l-1} > 100$

Now subtract $\frac{1}{4}$

$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} - \frac{1}{2} + \frac{1}{2(k+1)-1} + \cdots + \frac{1}{2l-1} - \frac{1}{4} > 99.7$

Fact: Conditionally convergent sums can be rearranged to converge to any real number.